

KK-EQUIVALENCE FOR AMALGAMATED FREE PRODUCT C*-ALGEBRAS

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ABSTRACT. We prove that *any* reduced amalgamated free product C*-algebra is *KK*-equivalent to the corresponding full amalgamated free product C*-algebra. The main ingredient of its proof is Julg–Valette’s geometric construction of Fredholm modules with Connes’s view for representation theory of operator algebras.

1. INTRODUCTION

In [2][3] Cuntz gave a strategy of computing the *K*-theory of the reduced C*-algebra $C_{\text{red}}^*(\Gamma)$ of a given discrete group Γ . The strategy consists of two parts:

- (1) proving that the canonical surjection $\lambda : C^*(\Gamma) \rightarrow C_{\text{red}}^*(\Gamma)$ (where $C^*(\Gamma)$ denotes the full C*-algebra of Γ) gives a *KK*-equivalence, that is, has an inverse in *KK*-theory, and
- (2) computing the *K*-theory of $C^*(\Gamma)$.

In fact, usual computations in *K*-theory are made by establishing suitable exact sequences, and the full group C*-algebra $C^*(\Gamma)$ is easier to handle than the reduced one $C_{\text{red}}^*(\Gamma)$. By the strategy, Cuntz indeed gave a much simpler proof of celebrated Pimsner–Voiculescu’s result of the *K*-theory of $C_{\text{red}}^*(\mathbb{F}_n)$ ([15]). Then Julg and Valette [11] achieved part (1) of the strategy when Γ acts on a tree with amenable stabilizers. In the direction, Pimsner gave in [13] an optimal result, but his strategy looks different from Cuntz’s one.

It is very natural (at least for us) to try to adapt Cuntz’s strategy to amalgamated free product C*-algebras. Part (2) of the strategy was achieved by Thomsen [17] under a very weak assumption. Thus, our main problem is part (1) of the strategy. It was Germain [5][6] who first tried to examine the strategy for plain free product C*-algebras, and he obtained the desired *KK*-equivalence result for plain free product C*-algebras of nuclear C*-algebras. Following Germain’s idea in [5][7] we recently proved in [9] (also see [8]) that the canonical surjection onto a given reduced amalgamated free product C*-algebra from the corresponding full one gives a *KK*-equivalence under the assumption that every free component is “strongly relative nuclear” against the amalgamated subalgebra. This was a byproduct of our attempt to seek for a suitable formulation of “relative nuclearity” for inclusions of C*-algebras.

In this paper we adapt, unlike [5][7][8][9], Julg–Valette’s geometric idea to the problem, and establish the optimal *KK*-equivalence result for amalgamated free product C*-algebras. We emphasize that the core part of the proof is very simple and just 3 pages long (though this paper is rather self-contained). To state our main result precisely, let us give a few terminologies. Let $\{(B \subset A_i, E_B^{A_i})\}_{i \in \mathcal{I}}$ be a countable family of quasi-unital inclusions of separable C*-algebras with nondegenerate conditional expectations from A_i onto B . Here $B \subset A_i$ is quasi-unital if BA_iB is norm-dense in A_i , and also $E_B^{A_i} : A_i \rightarrow B$ is nondegenerate if the associated GNS representation is faithful. Let $(A, E) = \star_{B, i \in \mathcal{I}} (A_i, E_i^{A_i})$ be the reduced amalgamated free product and we call A the reduced amalgamated free product C*-algebra. Also, let $\mathfrak{A} = \star_{B, i \in \mathcal{I}} A_i$ be the full amalgamated free product C*-algebra. With the notation we will prove the following:

Key words and phrases. amalgamated free product, *KK*-theory.

Theorem A. *The canonical surjection $\lambda : \mathfrak{A} \rightarrow A$ gives a KK -equivalence without any extra assumption.*

The proof is done by translating the “geometric” construction of Fredholm modules due to Julg–Valette (and its quantum group analog due to Vergnoux [20]) into a C^* -algebraic language following Connes’s view of correspondences. This is similar to our previous work [9] on relative nuclearity. More precisely, we will easily prove that the canonical surjection λ gives a KK -subequivalence like Julg–Valette [11] and Vergnoux [20]. Then we will directly prove that λ indeed gives a KK -equivalence. The latter is unnecessary in the amenable (quantum) group case [11][20] thanks to the existence of counits, and is the most original part of the present paper. As a bonus of the present approach we obtain the following corollary:

Corollary B. *Both \mathfrak{A} and A are K -nuclear if all the A_i are nuclear.*

Throughout this paper, we employ the following standard notation: For a Hilbert space H , we denote by $\mathbb{B}(H)$ the bounded linear operators on H and by $\mathbb{K}(H)$ the compact ones on H . For C^* -algebras A and B , $A \otimes B$ stands for the minimal tensor product. We use the symbol \odot for algebraic tensor products. For a subset \mathcal{S} of a normed space X , we denote by $[\mathcal{S}]$ the closed linear subspace of X generated by \mathcal{S} .

2. PRELIMINARIES

2.1. C^* -correspondences. For the theory of Hilbert C^* -modules, we refer to Lance’s book [12]. Let A and B be C^* -algebras. An A - B C^* -correspondence is a pair (X, π_X) such that X is a Hilbert B -module and π_X is a $*$ -homomorphism from A into the C^* -algebra $\mathbb{L}_B(X)$ of right B -linear adjointable operators on X . We denote by $\mathbb{K}_B(X)$ the C^* -ideal of $\mathbb{L}_B(X)$ generated by “rank one operators” $\theta_{\xi, \eta}$, $\xi, \eta \in X$ defined by $\theta_{\xi, \eta}(\zeta) := \xi \langle \eta, \zeta \rangle$. The *identity C^* -correspondence over A* is the pair (A, λ_A) , where A is equipped with the A -valued inner product $\langle x, y \rangle = x^*y$ for $x, y \in A$ and $\lambda_A : A \rightarrow \mathbb{L}_A(A)$ is defined by the left multiplication. It is known that $\mathbb{L}_A(A)$ is naturally isomorphic to the multiplier algebra $\mathcal{M}(A)$ of A .

We use the following two notions of tensor products for Hilbert C^* -modules. Let X be a Hilbert B -module and (Y, ϕ) be a B - C C^* -correspondence. We denote by $X \otimes_\phi Y$ the *interior tensor product* of X and (Y, ϕ) , which is given by separation and completion of $X \odot Y$ with respect to the C -valued inner product $\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \eta, \phi(\langle \xi, \xi' \rangle) \eta' \rangle$. There is a canonical map $\mathbb{L}_B(X) \rightarrow \mathbb{L}_C(X \otimes_\phi Y)$ sending T to the operator $T \otimes_\phi 1_Y : \xi \otimes \eta \mapsto (T\xi) \otimes \eta$. For a given $*$ -homomorphism $\pi_X : A \rightarrow \mathbb{L}_B(X)$ we define $\pi_X \otimes_\phi 1_Y : A \rightarrow \mathbb{L}_C(X \otimes_\phi Y)$ by $(\pi_X \otimes_\phi 1_Y)(a) = \pi_X(a) \otimes_\phi 1_Y$. When no confusion may arise, we use the notations $X \otimes_B Y$ and $\pi_X \otimes_B 1_Y$ for short.

For a Hilbert D -module Z , we denote by $X \otimes Z$ the *exterior tensor product* of X and Y , which is the completion of $X \odot Y$ with respect to the $B \otimes D$ -valued inner product $\langle \xi \otimes \zeta, \xi' \otimes \zeta' \rangle = \langle \xi, \xi' \rangle \otimes \langle \zeta, \zeta' \rangle$. When (Z, π_Z) is a C - D C^* -correspondence, there is a natural $*$ -homomorphism $\pi_X \otimes \pi_Z : A \otimes C \rightarrow \mathbb{L}_{B \otimes D}(X \otimes Z)$ so that $(X \otimes Z, \pi_X \otimes \pi_Z)$ is an $A \otimes C$ - $B \otimes D$ C^* -correspondence.

Let $B \subset A$ be a quasi-unital inclusion of C^* -algebras (i.e., BAB is norm-dense in A) with conditional expectation $E : A \rightarrow B$. We denote by $L^2(A, E)$ the Hilbert B -module given by separation and completion of A with respect to the B -valued inner product $\langle x, y \rangle = E(x^*y)$ for $x, y \in A$, and by $\pi_E : A \rightarrow \mathbb{L}_B(L^2(A, E))$ the $*$ -homomorphism induced from the left multiplication. The conditional expectation E is said to be *nondegenerate* if π_E is faithful. We denote by 1_A the unit of the multiplier algebra of A . Since the inclusion $B \subset A$ is quasi-unital, B contains an approximate unit for A . In particular, A is unital if and only if so is B , and they should have a common unit. Thus, we can uniquely extend E to a conditional expectation $\tilde{E} : A + \mathbb{C}1_A \rightarrow B + \mathbb{C}1_A$. Let ξ_E be the vector in $L^2(A + \mathbb{C}1_A, \tilde{E})$ corresponding to 1_A . We always identify $L^2(A, E)$ with $[\pi_{\tilde{E}}(A)\xi_E] \subset L^2(A + \mathbb{C}1_A, \tilde{E})$ and call the triple $(L^2(A, E), \pi_E, \xi_E)$ the *GNS-representation* associated with the conditional expectation E . Notice that ξ_E need not to be in $L^2(A, E)$ when A is non-unital.

2.2. KK -theory. Throughout this subsection, all C^* -algebras are assumed to be separable for simplicity. We refer the reader to [1] for KK -theory.

Definition 2.1. For (trivially graded) C^* -algebras A and B , a *Kasparov A - B bimodule* is a triple (X, ϕ, F) such that X is a countably generated graded Hilbert B -module, $\phi : A \rightarrow \mathbb{L}_B(X)$ is a $*$ -homomorphism of degree 0, and $F \in \mathbb{L}_B(X)$ is of degree 1 and satisfies the following condition:

- $[F, \phi(a)] \in \mathbb{K}_B(X)$ for $a \in A$,
- $(F - F^*)\phi(a) \in \mathbb{K}_B(X)$ for $a \in A$,
- $(1 - F^2)\phi(a) \in \mathbb{K}_B(X)$ for $a \in A$.

When $[F, \phi(a)] = (F - F^*)\phi(a) = (1 - F^2)\phi(a) = 0$ holds for every $a \in A$, we say that (X, ϕ, F) is *degenerate*. We denote by $\mathbb{E}(A, B)$ and $\mathbb{D}(A, B)$ the corrections of Kasparov A - B bimodules and degenerate ones, respectively.

We say that two Kasparov A - B bimodules (X, ϕ, F) and (Y, ψ, G) are *unitarily equivalent*, denoted by $(X, \phi, F) \cong (Y, \psi, G)$, if there exists a unitary $U \in \mathbb{L}(X, Y)$ of degree 0 such that $\psi = \text{Ad } U \circ \phi$ and $G = UFU^*$.

For any Hilbert B -module X , we set $IX := C([0, 1]) \otimes X$. In particular, we set $IB = C([0, 1]) \otimes B$. For each $t \in [0, 1]$ we still denote by t the surjective $*$ -homomorphism $IB \cong C([0, 1], B) \ni f \mapsto f(t) \in B$. Note that we have a natural isomorphism $IX \otimes_t B \cong X$ for every $t \in [0, 1]$.

Definition 2.2. Two Kasparov A - B bimodules (X_0, ϕ_0, F_0) and (X_1, ϕ_1, F_1) are said to be *homotopic* if there exists a Kasparov A - IB bimodule (Y, ψ, G) such that $(Y \otimes_t B, \psi \otimes_t 1_B, G \otimes_t 1_B) \cong (X_t, \phi_t, F_t)$ for $t = 0, 1$. The KK -group $KK(A, B)$ is the set of homotopy equivalence classes of all Kasparov A - B bimodules.

The next technical lemma will be used later.

Lemma 2.3. *Let P, Q and R be separable C^* -algebras and let $(X, \psi_i, F) \in \mathbb{E}(Q, R)$ be given for $i = 0, 1$. Suppose that there exist a surjective $*$ -homomorphism $\pi : P \rightarrow Q$ and a family of Kasparov P - R bimodules (X, ϕ_t, F) for $t \in [0, 1]$ satisfying*

- (i) *the mapping $[0, 1] \ni t \mapsto \phi_t(a)$ is strictly continuous for each $a \in P$;*
- (ii) *ϕ_t factors through $\pi : P \rightarrow Q$ for every $t \in [0, 1]$;*
- (iii) *$\phi_i = \psi_i \circ \pi$ holds for $i = 0, 1$.*

Then, (X, ψ_0, F) and (X, ψ_1, F) are homotopic.

Proof. By assumption, there exists a $*$ -homomorphism $\phi : P \rightarrow \mathbb{L}_{IR}(IX)$ such that $(IX, \phi, F \otimes 1_{C([0, 1])}) \in \mathbb{E}(P, IR)$ and $\phi \otimes_t 1_R = \phi_t$ for $t \in [0, 1]$. Since one has $\|\phi(a)\| = \sup_{0 \leq t \leq 1} \|\phi_t(a)\| \leq \|\pi(a)\|$ for $a \in P$, there exists $\psi : Q \rightarrow \mathbb{L}_{IR}(IX)$ such that $\phi = \psi \circ \pi$. We then have $(IX, \psi, F \otimes 1_{C([0, 1])}) \in \mathbb{E}(Q, IR)$ and the evaluations of this Kasparov bimodule at endpoints are exactly (X, ψ_i, F) , $i = 0, 1$. \square

The KK -group becomes an additive group in the following way: For $\alpha, \beta \in KK(A, B)$ implemented by $(X, \phi, F), (Y, \psi, G)$, respectively, $\alpha + \beta$ is the element implemented by $(X \oplus Y, \phi \oplus \psi, F \oplus G)$. All degenerate Kasparov bimodules are homotopic to the trivial bimodule $0 = (0, 0, 0)$ and define the zero element in $KK(A, B)$. Let X_0 and X_1 be the even and odd parts of X so that $X = X_0 \oplus X_1$ and let $-X$ be the graded Hilbert B -module with the even part X_1 and the odd part X_0 . The inverse of α is implemented by $(-X, \text{Ad } U \circ \phi, UFU^*)$, where $U : X \rightarrow -X$ is the natural unitary.

For any $*$ -homomorphism $\phi : A \rightarrow B$, we have $(B \oplus 0, \phi \oplus 0, 0) \in \mathbb{E}(A, B)$ and still denote by ϕ the corresponding element in $KK(A, B)$.

For $\alpha \in KK(A, B)$ and $\gamma \in KK(B, C)$, the *Kasparov product* of α and γ is denoted by $\gamma \circ \alpha$ (or $\alpha \otimes_B \gamma$). When one of α and β comes from a $*$ -homomorphism, the construction of the Kasparov product is very simple (and we will use Kasparov products only in these special cases). Indeed, if γ comes from a $*$ -homomorphism $\gamma : B \rightarrow C$ with $[\gamma(B)C] = C$ and α is implemented

by (X, ϕ, F) , then the Kasparov product $\gamma \circ \alpha$ is implemented by $(X \otimes_\gamma C, \phi \otimes_\gamma 1_C, F \otimes_\gamma 1_C)$. Similarly, when α is a $*$ -homomorphism from A into B and γ is implemented by (Y, ψ, G) with $[\psi(B)Y] = Y$, the Kasparov product $\gamma \circ \alpha$ is implemented by $(Y, \psi \circ \alpha, G)$.

Definition 2.4. An element $\alpha \in KK(A, B)$ is said to be a *KK-equivalence* if there exists $\beta \in KK(B, A)$ such that $\text{id}_A = \beta \circ \alpha$ and $\text{id}_B = \alpha \circ \beta$. In this case, A and B are said to be *KK-equivalent*.

Note that *KK-equivalence* between A and B implies $KK(A, C) \cong KK(B, C)$ and $KK(C, A) \cong KK(C, B)$ for any separable C^* -algebra C .

Finally, we recall the notion of *K-nuclearity* in the sense of Skandalis [16].

Theorem 2.5 ([16, Theoreme 1.5]). *Let A and B be separable C^* -algebras and let $\pi : A \rightarrow \mathbb{B}(H)$ be a faithful and essential representation on a separable Hilbert space H . For a given A - B C^* -correspondence (X, σ) with X countably generated, the following are equivalent:*

- (i) *For any unit vector $\xi \in X$ the c.c.p. (completely contractive positive) map $A \ni a \mapsto \langle \xi, \sigma(a)\xi \rangle \in B$ is nuclear.*
- (ii) *For any $x \in \mathbb{K}_B(X)$ of norm 1, the c.c.p. map $A \ni a \mapsto x^* \sigma(a)x \in \mathbb{K}_B(X)$ is nuclear.*
- (iii) *There exists a sequence of isometries $V_n \in \mathbb{L}_B(X, H \otimes B)$ such that $\sigma(a) - V_n^*(\pi(a) \otimes 1_A)V_n \in \mathbb{K}_B(X)$ and $\lim_{n \rightarrow \infty} \|\sigma(a) - V_n^*(\pi(a) \otimes 1_A)V_n\| = 0$ for all $a \in A$.*

When any of these three conditions holds, we say that (X, σ) is nuclear.

Note that any C^* -correspondence of the form $(X \otimes_B Y, \pi_X \otimes_B 1_Y)$ is nuclear whenever B is nuclear (see e.g. [9, Remark 2.11]).

Definition 2.6. A separable C^* -algebra A is said to be *K-nuclear* if id_A in $KK(A, A)$ is implemented by a Kasparov bimodule (X, ϕ, F) such that (X, ϕ) is nuclear.

2.3. Amalgamated free products. Let $\{B \subset A_i\}_{i \in \mathcal{I}}$ be a family of inclusions of C^* -algebras. Recall that the *full amalgamated free product* of $\{A_i\}_{i \in \mathcal{I}}$ over B is a C^* -algebra \mathfrak{A} generated by the images of injective $*$ -homomorphisms $f_i : A_i \hookrightarrow \mathfrak{A}$ such that $f_i|_B = f_j|_B$ for $i, j \in \mathcal{I}$ and satisfying the following universal property: for any C^* -algebra C and $*$ -homomorphisms $\pi_i : A_i \rightarrow C$ satisfying $\pi_i|_B = \pi_j|_B$ for $i, j \in \mathcal{I}$, there exists a unique $*$ -homomorphism $\star_{i \in \mathcal{I}} \pi_i : \mathfrak{A} \rightarrow C$ such that $(\star_{i \in \mathcal{I}} \pi_i) \circ f_i = \pi_i$ for $i \in \mathcal{I}$. Since the full amalgamated free product is unique up to isomorphism, we denote it by $\star_{B, i \in \mathcal{I}} A_i$. We identify A_i with $f_i(A_i)$ so that $A_i \subset \star_{B, i \in \mathcal{I}} A_i$ for every $i \in \mathcal{I}$.

Further assume that, the inclusion $B \subset A_i$ is quasi-unital and there exists a nondegenerate conditional expectation $E_B^{A_i} : A_i \rightarrow B$ for each $i \in \mathcal{I}$. In [21], Voiculescu introduced reduced amalgamated free products of unital inclusions of C^* -algebras with conditional expectations. To reduce Theorem A to the case when $\mathcal{I} = \{1, 2\}$, we need to extend Voiculescu's definition to quasi-unital inclusions. For any $m \in \mathbb{N}$, set $\mathcal{I}_m := \{\iota : \{1, \dots, m\} \rightarrow \mathcal{I} \mid \iota(k) \neq \iota(k+1) \text{ for } 1 \leq k \leq m-1\}$. Recall that the *reduced amalgamated free product* of $\{(A_i, E_B^{A_i})\}_{i \in \mathcal{I}}$ is a pair (A, E) such that

- A is a C^* -algebra generated by the images of injective $*$ -homomorphisms $g_i : A_i \hookrightarrow A$ such that $g_i|_B = g_j|_B$ for $i, j \in \mathcal{I}$;
- E is a nondegenerate conditional expectation from A onto $g_i(B)$ (independent of i);
- one has $E(g_{\iota(1)}(x_1)g_{\iota(2)}(x_2) \cdots g_{\iota(m)}(x_m)) = 0$ for any $m \geq 1$, $\iota \in \mathcal{I}_m$, and $x_k \in \ker E_B^{A_{\iota(k)}}$ for $1 \leq k \leq m$.

We will also identify A_i with $g_i(A_i)$ for every $i \in \mathcal{I}$. Since the pair (A, E) is determined by the above three conditions, we will write $\star_{B, \mathcal{I}}(A_i, E_B^{A_i}) := (A, E)$. Clearly, we have a canonical surjection $\lambda : \mathfrak{A} \rightarrow A$ satisfying that $\lambda \circ f_i = g_i$ for every $i \in \mathcal{I}$.

We recall the construction of (A, E) ([21]). Let (X_i, π_{X_i}, ξ_i) be the GNS-representation associated with $E_B^{A_i}$ (see §§2.1) and set $A_i^\circ := \ker E_B^{A_i}$ and $X_i^\circ = X_i \ominus \xi_i B = [\pi_{X_i}(A_i^\circ)\xi_i]$ for

$i \in \mathcal{I}$. Recall that the free product of $\{(X_i, \xi_i)\}_{i \in \mathcal{I}}$ is the Hilbert B -module (X, ξ_0) defined to be $\xi_0 B \oplus \bigoplus_{m \geq 1} \bigoplus_{\substack{\iota \in \mathcal{I}_m \\ \iota(1) \neq i}} X_{\iota(1)}^\circ \otimes_B \cdots \otimes_B X_{\iota(m)}^\circ$, where we define $\langle \xi_0 b, \xi_0 c \rangle = b^* c$ for $b, c \in B$. We identify X_i with $\xi_0 B \oplus X_i^\circ$ so that $X_i \subset X$. For each $i \in \mathcal{I}$, we consider the following submodules:

$$X(\ell, i) := \xi_0 B \oplus \bigoplus_{m \geq 1} \bigoplus_{\substack{\iota \in \mathcal{I}_m \\ \iota(1) \neq i}} X_{\iota(1)}^\circ \otimes_B \cdots \otimes_B X_{\iota(m)}^\circ,$$

$$X(r, i) := \xi_0 B \oplus \bigoplus_{m \geq 1} \bigoplus_{\substack{\iota \in \mathcal{I}_m \\ \iota(m) \neq i}} X_{\iota(1)}^\circ \otimes_B \cdots \otimes_B X_{\iota(m)}^\circ.$$

Then, there is a natural unitary $V_i : X \cong X_i \otimes_B X(\ell, i)$ (see [21]). We set $g_i := \text{Ad } V_i \circ (\pi_{X_i} \otimes_B 1_{X(\ell, i)})$ and $A = C^*(g_i(A_i) \mid i \in \mathcal{I})$. Then, the g_i 's agree with each other on B and the compression map to $\xi_0 B$ gives the desired conditional expectation $E : A \rightarrow B$. Note that the representation $A \subset \mathbb{L}_B(X)$ with ξ_0 is nothing but the GNS-representation associated with E , and hence E must be nondegenerate.

The following lemma is probably well-known, but we give its proof for the reader's convenience.

Lemma 2.7. *Let $\mathfrak{A} = \star_{B, i \in \mathcal{I}} A_i$ and $(A, E) = \star_{B, i \in \mathcal{I}} (A_i, E_B^{A_i})$ be as above. Let C be a C^* -algebra and (Z, π_Z) be an \mathfrak{A} - C C^* -correspondence. Suppose that for each $i \in \mathcal{I}$ there exists a subset $\mathcal{S}_i \subset A_i^\circ$ generating A_i° as a normed space, and there exists a cyclic subspace $\Gamma \subset Z$ (i.e., $[\pi_Z(\mathfrak{A})\Gamma C] = Z$ holds) which satisfies the freeness condition: for any $m \in \mathbb{N}$, $\iota \in \mathcal{I}_m$, $x_k \in \mathcal{S}_{\iota(k)}$ for $1 \leq k \leq m$ and $\xi, \eta \in \Gamma$, one has $\langle \xi, \pi_Z(x_1 x_2 \cdots x_m) \eta \rangle = 0$. Then, π_Z factors through $\lambda : \mathfrak{A} \rightarrow A$.*

Proof. We will show that $\ker \lambda \subset \ker \pi_Z$. Choose and fix $z \in \ker \lambda$ arbitrarily. By assumption, it suffices to show that $\langle \xi, \pi_Z(xzy) \eta \rangle = 0$ for all $x, y \in \mathfrak{A}$ and $\xi, \eta \in \Gamma$. We may assume that x and y are in $*\text{-alg}(\bigcup_{i \in \mathcal{I}} A_i)$. Take a sequence z_n in $*\text{-alg}(\bigcup_{i \in \mathcal{I}} A_i)$ such that $\lim_{n \rightarrow \infty} \|z - z_n\| = 0$. For each $n \geq 1$ there exists $b_n \in B$ such that $xz_n y - b_n$ is a sum of some elements of the form $x_1 \cdots x_m$ for some $m \geq 1$, $\iota \in \mathcal{I}_m$ and $x_k \in A_{\iota(k)}^\circ$ for $1 \leq k \leq m$ so that $b_n = E(xz_n y)$. The assumption on Γ implies that $\langle \xi, \pi_Z(xz_n y - b_n) \eta \rangle = 0$, and hence we have $\|\langle \xi, \pi_Z(xzy) \eta \rangle\| = \lim_{n \rightarrow \infty} \|\langle \xi, \pi_Z(b_n) \eta \rangle\| \leq \limsup_{n \rightarrow \infty} \|\xi\| \|\eta\| \|E(xz_n y)\| = 0$. \square

3. PROOF

3.1. Case of two free components. We first deal with the case when $\mathcal{I} = \{1, 2\}$. Let $(A, E) = (A_1, E_B^{A_1}) \star_B (A_2, E_B^{A_2})$, $\mathfrak{A} = A_1 \star_B A_2$ and $\lambda : \mathfrak{A} \rightarrow A$ be as in Theorem A. As in the previous section, let (X, π_X, ξ_0) be the GNS-representation associated with E and identify $X_i := L^2(A_i, E_B^{A_i})$ with $\xi_0 B \oplus X_i^\circ$ for $i = 1, 2$. Let $E_{A_i} : A \rightarrow A_i$ be the canonical conditional expectation given by the compression map to X_i and let (Y_i, π_{Y_i}, η_i) be the GNS-representation associated with E_{A_i} for $i = 1, 2$. Note that any vector of the form $\xi_0 \otimes a$ with $a \in A_i$ sits in $X(r, i) \otimes_B A_i$ for each $i \in \mathcal{I}$ thanks to the assumption that $B \subset A$ is quasi-unital. The following lemma can be shown in the same manner as [20, Lemma 3.1], but we give its proof for the reader's convenience.

Lemma 3.1. *The exists a unitary $S_i : X(r, i) \otimes_B A_i \rightarrow Y_i$ satisfying that $S_i(\xi_0 \otimes y) = \eta_i y$ and $S_i(x_1 \cdots x_m \xi_0 \otimes y) = x_1 \cdots x_m \eta_i y$ for all $y \in A_i$ and $m \in \mathbb{N}$, $\iota \in \mathcal{I}_m$ with $\iota(m) \neq i$, and $x_k \in A_{\iota(k)}^\circ$ for $1 \leq k \leq m$.*

Proof. Note that if S_i is bounded, then it must be surjective. Thus, it suffices to show that S_i is an isometry. By the polarization trick, we only have to verify that $E_{A_i}(x^* x) = E(x^* x)$ for all $x = x_1 \cdots x_m$ with $m \in \mathbb{N}$, $\iota \in \mathcal{I}_m$, $\iota(m) \neq i$ and $x_k \in A_{\iota(k)}^\circ$ for $1 \leq k \leq m$. When $m = 1$, this follows from the fact that E_{A_i} is given by the compression to X_i . Assume that we have shown for $k = 1, \dots, m$. For $\iota \in \mathcal{I}_{m+1}$ with $\iota(m+1) \neq i$, take $x_k \in A_{\iota(k)}^\circ$ arbitrarily. If we put $y = x_2 \cdots x_{m+1}$

and $b = E(x_1^* x_1)$, then the induction hypothesis implies that $E_{A_i}(y^* b y) = E(y^* b y)$. Thus, we have $E_{A_i}(x^* x) = E_{A_i}(y^* E(x_1^* x_1) y) + E_{A_i}(y^* (x_1^* x_1 - E(x_1^* x_1)) y) = E_{A_i}(y^* b y) = E(y^* b y) = E(y^* E(x_1^* x_1) y) + E(y^* (x_1^* x_1 - E(x_1^* x_1)) y) = E(x^* x)$. Hence, we are done. \square

Consider two A - \mathfrak{A} C^* -correspondences $(Z^+, \pi^+) := (X \otimes_B \mathfrak{A}, \pi_X \otimes_B 1)$ and $(Z^-, \pi^-) := \bigoplus_{i=1}^2 (Y_i \otimes_{A_i} \mathfrak{A}, \pi_{Y_i} \otimes_{A_i} 1)$. Notice that the vector $\zeta_i := \eta_i \otimes 1_{\mathfrak{A}}$ is not necessarily in Z^- , but one has $\zeta_i \mathfrak{A} \subset Z^-$. Define the isometry $S : Z^+ \rightarrow Z^-$ by

$$\begin{cases} S_1 \otimes_{A_1} 1 : X(r, 1)^\circ \otimes_B \mathfrak{A} \rightarrow Y_1^\circ \otimes_{A_1} \mathfrak{A}; \\ S_2 \otimes_{A_2} 1 : X(r, 2) \otimes_B \mathfrak{A} \rightarrow Y_2 \otimes_{A_2} \mathfrak{A}. \end{cases}$$

Lemma 3.2 (c.f. [20, Theorem 3.3 (2)]). *The operator S satisfies that $\ker S^* = \zeta_1 \mathfrak{A}$ and $\pi^-(a)S - S\pi^+(a)$ is compact for all $a \in A$. Consequently, the triple $(Z^+ \oplus Z^-, \pi^+ \oplus \pi^-, \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix})$ is an A - \mathfrak{A} Kasparov bimodule.*

Proof. The first assertion is obvious. Thus, it suffices to show $\pi^-(x)S - S\pi^+(x)$ is compact for all $x \in A_1 \cup A_2$. In fact, since each $x \in A_1$ enjoys $xX(r, 1)^\circ \subset X(r, 1)^\circ$ and $xX(r, 2) \subset X(r, 2)$, one has $\pi^-(x)S = S\pi^+(x)$ for $x \in A_1$. If we define $S' : Z^+ \rightarrow Z^-$ by $S'\xi_0 \otimes a = \zeta_1 a$ for $a \in \mathfrak{A}$ and by S on $X^\circ \otimes_B \mathfrak{A}$, then S' intertwines the actions of A_2 by the above argument. Since S is a compact perturbation of S' , we are done. \square

Remark 3.3. Recall that the Bass–Serre tree associated with an amalgamated free product group $G = G_1 \star_H G_2$ is the graph whose vertex and edge sets are given by $\Delta^0 = G/G_1 \sqcup G/G_2$ and $\Delta^1 = G/H$, respectively. Consider the unitary representations of G on $\ell^2(\Delta^0)$ and $\ell^2(\Delta^1)$ induced from the action of G on (Δ^0, Δ^1) . In [9], we saw that C^* -correspondences play a role of unitary representations for groups. In our theory, the canonical representation G on $\ell^2(G/H)$ corresponds to $(L^2(A, E) \otimes_B \mathfrak{A}, \pi_E \circ \lambda \otimes_B 1)$ (c.f. [10]). Thus, the C^* -correspondences $(Z^+, \pi^+ \circ \lambda)$ and $(Z^-, \pi^- \circ \lambda)$ should play a role of the Bass–Serre tree in C^* -algebra theory. Also, the adjoint of S corresponds to the co-isometry of Julg–Valette in [11].

Here is the main technical result of this paper.

Theorem 3.4. *With the notation above, let α be the element in $KK(A, \mathfrak{A})$ implemented by $(Z^+ \oplus Z^-, \pi^+ \oplus \pi^-, \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix})$. Then, we have $\alpha \circ \lambda + \text{id}_{\mathfrak{A}} = 0$ and $\lambda \circ \alpha + \text{id}_A = 0$.*

Proof. We first prove that $\alpha \circ \lambda + \text{id}_{\mathfrak{A}} = 0$ following the proof of [20, Theorem 3.3 (3)]. Set $\rho^+ := \pi^+ \circ \lambda$ and $\rho^- := \pi^- \circ \lambda$. Define the unitary $U : Z^+ \oplus \mathfrak{A} \rightarrow Z^-$ by S on Z^+ and by $U(0 \oplus a) := \zeta_1 a$ for $a \in \mathfrak{A}$. Since S is a compact perturbation of U , $\alpha \circ \lambda + \text{id}_{\mathfrak{A}}$ is implemented by $((Z^+ \oplus \mathfrak{A}) \oplus Z^-, (\rho^+ \oplus \lambda_{\mathfrak{A}}) \oplus \rho^-, \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix})$ (see §§2.2). Take a norm continuous path $(v_t)_{0 \leq t \leq 1}$ of unitaries in $\mathbb{M}_2(\mathbb{C})$ such that $v_0 = 1$ and $v_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. With the natural identification $\mathbb{M}_2(\mathbb{C}) \subset \mathbb{M}_2(\mathbb{C}) \otimes \mathcal{M}(\mathfrak{A}) = \mathbb{L}_{\mathfrak{A}}(\zeta_1 \mathfrak{A} \oplus \zeta_2 \mathfrak{A})$ we define the unitary $u_t \in \mathbb{L}_{\mathfrak{A}}(Z^-)$ by v_t on $\zeta_1 \mathfrak{A} \oplus \zeta_2 \mathfrak{A}$ and by the identity operator on $Z^- \ominus (\zeta_1 \mathfrak{A} \oplus \zeta_2 \mathfrak{A})$. Since the restriction of B to $\zeta_1 \mathfrak{A} \oplus \zeta_2 \mathfrak{A}$ is just $\mathbb{C}1 \otimes B \subset \mathbb{M}_2(\mathbb{C}) \otimes \mathcal{M}(A)$ with the above identification, the family $(u_t)_{0 \leq t \leq 1}$ forms a norm continuous path of unitaries in $\pi^-(B)' \cap (\mathbb{C}1 + \mathbb{K}_B(Z^-))$ satisfying that $u_0 = 1$ and u_1 switches $\zeta_1 a$ and $\zeta_2 a$ for each $a \in \mathfrak{A}$. Let $\iota_i : A_i \hookrightarrow A$ be the inclusion map for $i = 1, 2$. Since $\pi^- \circ \iota_1$ agrees with $\text{Ad } u_t \circ \pi^- \circ \iota_2$ on B , we have the natural $*$ -homomorphism $\phi_t := (\pi^- \circ \iota_1) \star (\text{Ad } u_t \circ \pi^- \circ \iota_2) : \mathfrak{A} \rightarrow \mathbb{L}_{\mathfrak{A}}(Z^-)$ thanks to the universality of \mathfrak{A} . Then, the Kasparov bimodules

$$((Z^+ \oplus \mathfrak{A}) \oplus Z^-, (\rho^+ \oplus \lambda_{\mathfrak{A}}) \oplus \phi_t, \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}), \quad t \in [0, 1]$$

satisfy conditions (i) and (ii) in Lemma 2.3 (with $P = Q = \mathfrak{A}$), and its evaluation at $t = 0$ implements $\alpha \circ \lambda + \text{id}_{\mathfrak{A}}$. Thus, we need to show that $((Z^+ \oplus \mathfrak{A}) \oplus Z^-, (\rho^+ \oplus \lambda_{\mathfrak{A}}) \oplus \phi_1, \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix})$ is degenerate, that is,

$$U(\rho^+(x) \oplus \lambda_{\mathfrak{A}}(x)) = \phi_1(x)U \quad \text{for } x \in \mathfrak{A}. \quad (1)$$

Since U is unitary, we may assume that x is in $A_1 \cup A_2^\circ$. When x is in A_1 , the above equation is trivial because S intertwines $\pi^+(x)$ and $\pi^-(x)$. Let S' be as in the proof of the previous lemma.

Then, we have $u_1^*U = S'$ on Z^+ and $u_1^*U(0 \oplus a) = \zeta_2 a$ for $a \in \mathfrak{A}$. Since S' intertwines the actions of A_2 , we have $U(\pi^+(x) \oplus \lambda_{\mathfrak{A}}(x)) = u_1 \pi^-(x) u_1^*U$ for every $x \in A_2$. Thus we obtain equation (1), and hence Lemma 2.3 shows $\alpha \circ \lambda + \text{id}_{\mathfrak{A}} = 0$.

We next prove that $\lambda \circ \alpha + \text{id}_A = 0$ in $KK(A, A)$. Note that $\lambda \circ \alpha + \text{id}_A$ is implemented by the Kasparov A - A bimodule

$$\left(((Z^+ \otimes_{\lambda} A) \oplus A) \oplus (Z^- \otimes_{\lambda} A), ((\pi^+ \otimes_{\lambda} 1_A) \oplus \lambda_A) \oplus (\pi^- \otimes_{\lambda} 1_A), \begin{bmatrix} 0 & U \otimes_{\lambda} 1_A \\ U^* \otimes_{\lambda} 1_A & 0 \end{bmatrix} \right)$$

(see §2.2). We observe that the family of Kasparov \mathfrak{A} - A bimodules

$$\left(((Z^+ \otimes_{\lambda} A) \oplus A) \oplus (Z^- \otimes_{\lambda} A), ((\rho^+ \otimes_{\lambda} 1_A) \oplus \lambda) \oplus (\phi_t \otimes_{\lambda} 1_A), \begin{bmatrix} 0 & U \otimes_{\lambda} 1 \\ U^* \otimes_{\lambda} 1 & 0 \end{bmatrix} \right), \quad t \in [0, 1]$$

satisfy conditions (i) and (ii) in Lemma 2.3 (with $P = \mathfrak{A}$ and $Q = A$) and its evaluations at endpoints implement $(\lambda \circ \alpha + \text{id}_A) \circ \lambda$ and 0. Thus, by Lemma 2.3 and the fact that $\pi_X : A \rightarrow \mathbb{L}_B(X)$ is faithful, it suffices to show that $\phi_t \otimes_{\pi_X \circ \lambda} 1_X : \mathfrak{A} \rightarrow \mathbb{L}_B(Z^- \otimes_{\pi_X \circ \lambda} X)$ factors through $\lambda : \mathfrak{A} \rightarrow A$ for every $t \in [0, 1]$. If we set $\sigma := \bigoplus_{i=1}^2 \pi_{Y_i} \otimes_{A_i} 1_X : A \rightarrow \mathbb{L}_B((Y_1 \otimes_{A_1} X) \oplus (Y_2 \otimes_{A_2} X))$ and $w_t := u_t \otimes_{\pi_X \circ \lambda} 1_X \in \mathbb{L}_B((Y_1 \otimes_{A_1} X) \oplus (Y_2 \otimes_{A_2} X))$, then $\phi_t \otimes_{\pi_X \circ \lambda} 1_X$ coincides with $\psi_t := (\sigma \circ \iota_1) \star (\text{Ad } w_t \circ \sigma \circ \iota_2)$. Note that $\psi_0 = \sigma \circ \lambda$ and $\psi_1 \cong (\rho^+ \otimes_B 1_X) \oplus \pi_X \circ \lambda$ apparently factor through λ . Thus, we only have to deal with $0 < t < 1$ and we write $w = w_t$ for short.

For the convenience, we identify $X(r, i) \otimes_B X$ with $Y_i \otimes_{A_i} X$ via $S_i \otimes_{A_i} 1$ as *right* B -modules. To distinguish between vectors in $X(r, 1) \otimes_B X$ and $X(r, 2) \otimes_B X$, we use the symbols $\dot{\otimes}$ and $\ddot{\otimes}$ as markers in such a way that, for $\zeta \in X$ we denote by $\xi_0 \dot{\otimes} \zeta \in X(r, 1) \otimes_B X$ and $\xi_0 \ddot{\otimes} \zeta \in X(r, 2) \otimes_B X$ the vectors corresponding to $\eta_1 \otimes \zeta$ and $\eta_2 \otimes \zeta$, respectively. Thanks to Lemma 2.7, the proof will be completed by proving the following claim:

Claim 3.5. *The subspace $\Gamma := w(\xi_0 B \dot{\otimes}_B X(\ell, 1)) + \xi_0 B \ddot{\otimes}_B X(\ell, 2)$ satisfies the assumption of Lemma 2.7.*

We first show that Γ is cyclic for $\psi_t(\mathfrak{A})$. Let $\Lambda := [\psi_t(\mathfrak{A})\Gamma]$. We set $\mathfrak{X}_0 := \xi_0 B$, $\mathfrak{X}_m := \bigotimes_{i \in \mathcal{I}_m} X_{\iota(i)}^{\circ} \otimes_B \cdots \otimes_B X_{\iota(m)}^{\circ}$, $\mathfrak{X}_m(\ell, i) = \mathfrak{X}_m \cap X(\ell, i)$ and $\mathfrak{X}_m(r, i) = \mathfrak{X}_m \cap X(r, i)$ for $m \in \mathbb{N}$ and $i = 1, 2$. It suffices to show that, for any $m \in \{0\} \cup \mathbb{N}$, Λ contains

$$\mathfrak{Y}_m := \left(\bigoplus_{k=0}^m \mathfrak{X}_k(r, 1) \dot{\otimes}_B \mathfrak{X}_{m-k} \right) \oplus \left(\bigoplus_{k=0}^m \mathfrak{X}_k(r, 2) \ddot{\otimes}_B \mathfrak{X}_{m-k} \right).$$

We will show this by induction. When $m = 0$, this is trivial because $\mathfrak{Y}_0 = (\xi_0 B \dot{\otimes}_B \xi_0 B) \oplus (\xi_0 B \ddot{\otimes}_B \xi_0 B) = w(\xi_0 B \dot{\otimes}_B \xi_0 B) + \xi_0 B \ddot{\otimes}_B \xi_0 B$. Suppose that Λ contains \mathfrak{Y}_k for $0 \leq k \leq m$. Since w is equal to 1 on the complement of $(\xi_0 B \dot{\otimes}_B X) \oplus (\xi_0 B \ddot{\otimes}_B X)$, it is easily seen that

$$\left(\bigoplus_{k=2}^{m+1} \mathfrak{X}_k(r, 1) \dot{\otimes}_B \mathfrak{X}_{m+1-k} \right) \oplus \left(\bigoplus_{k=2}^{m+1} \mathfrak{X}_k(r, 2) \ddot{\otimes}_B \mathfrak{X}_{m+1-k} \right) \subset \Lambda.$$

Thus, we only have to check that Λ contains the following six subspaces:

$$\begin{array}{lll} X_2^{\circ} \dot{\otimes}_B \mathfrak{X}_m, & \xi_0 B \dot{\otimes}_B \mathfrak{X}_{m+1}(\lambda, 1), & \xi_0 B \dot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 2), \\ X_1^{\circ} \ddot{\otimes}_B \mathfrak{X}_m, & \xi_0 B \ddot{\otimes}_B \mathfrak{X}_{m+1}(\lambda, 1), & \xi_0 B \ddot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 2). \end{array}$$

By assumption of induction, one has $w(\xi_0 B \dot{\otimes}_B \mathfrak{X}_m) \subset \mathfrak{Y}_m \subset \Lambda$, and hence $X_2^{\circ} \dot{\otimes}_B \mathfrak{X}_m = [w\sigma(A_2^{\circ})w w^*(\xi_0 B \dot{\otimes}_B \mathfrak{X}_m)] \subset \Lambda$. We also have $X_1^{\circ} \ddot{\otimes}_B \mathfrak{X}_m = [\sigma(A_1)(\xi_0 B \ddot{\otimes}_B \mathfrak{X}_m)] \subset \Lambda$. We observe that $w(\xi_0 B \ddot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 1)) = [w\sigma(A_2^{\circ})w^*w(\xi_0 B \ddot{\otimes}_B \mathfrak{X}_m(\ell, 2))] \subset \Lambda$ and $w(\xi_0 B \ddot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 1)) \subset \Gamma$ by the definition of Γ . Thus, one has

$$\xi_0 B \dot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 1) + \xi_0 B \ddot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 1) = w(\xi_0 B \dot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 1) + \xi_0 B \ddot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 1)) \subset \Lambda.$$

Finally we obtain that $\xi_0 B \ddot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 2) = [\sigma(A_1)(\xi_0 B \ddot{\otimes}_B \mathfrak{X}_m(\ell, 1))] \subset \Lambda$ and $\xi_0 B \ddot{\otimes}_B \mathfrak{X}_{m+1}(\ell, 2) \subset \Gamma$ by the definition of Γ again. Therefore, by induction, it follows that Γ is cyclic for $\psi_t(\mathfrak{A})$.

We next show that $\Gamma = w(\xi_0 B \dot{\otimes}_B X(\ell, 1)) + \xi_0 B \ddot{\otimes}_B X(\ell, 2)$ satisfies the freeness condition. Let $\Gamma_1 := \xi_0 B \dot{\otimes}_B X(\ell, 2)^\circ$ and $\Gamma_2 := w(\xi_0 B \ddot{\otimes}_B X(\ell, 1)^\circ)$. We then claim that the following inclusions hold:

$$\sigma(A_1^\circ)\Gamma \subset \Gamma_1 + X_1^\circ \ddot{\otimes}_B X \quad \text{and} \quad w\sigma(A_2^\circ)w^*\Gamma \subset \Gamma_2 + X_2^\circ \dot{\otimes}_B X. \quad (2)$$

Indeed, for any $x \in A_1^\circ$ one has

$$\begin{aligned} \sigma(x)w(\xi_0 B \dot{\otimes}_B X(\ell, 1)) &\subset \sigma(x)(\xi_0 B \dot{\otimes}_B X(\ell, 1) + \xi_0 B \ddot{\otimes}_B X(\ell, 1)) \\ &\subset \xi_0 B \dot{\otimes}_B X(\ell, 2)^\circ + X_1^\circ \ddot{\otimes}_B X(\ell, 1) \\ &\subset \Gamma_1 + X_1^\circ \ddot{\otimes}_B X \end{aligned}$$

and $\sigma(x)(\xi_0 B \ddot{\otimes}_B X(\ell, 2)) \subset X_1^\circ \ddot{\otimes}_B X(\ell, 2)$. Similarly, for any $y \in A_2^\circ$ one has

$$w\sigma(y)w^*w(\xi_0 B \dot{\otimes}_B X(\ell, 1)) \subset w(X_2^\circ \dot{\otimes}_B X(\ell, 1)) = X_2^\circ \dot{\otimes}_B X(\ell, 1)$$

and

$$\begin{aligned} w\sigma(y)w^*(\xi_0 B \ddot{\otimes}_B X(\ell, 2)) &\subset w\sigma(y)(\xi_0 B \dot{\otimes}_B X(\ell, 2) + \xi_0 B \ddot{\otimes}_B X(\ell, 2)) \\ &\subset w(X_2^\circ \dot{\otimes}_B X(\ell, 2)) + w(\xi_0 B \ddot{\otimes}_B X(\ell, 1)^\circ) \\ &= X_2^\circ \dot{\otimes}_B X(\ell, 2) + \Gamma_2. \end{aligned}$$

The subspaces on the right hand side in both equations (2) are apparently orthogonal to Γ , and one can easily verify that $\sigma(A_1^\circ)\Gamma_2 \subset \Gamma_1 + X_1^\circ \ddot{\otimes}_B X$ and $w\sigma(A_2^\circ)w^*\Gamma_1 \subset \Gamma_2 + X_2^\circ \dot{\otimes}_B X$. Since $w = 1$ on the complement of $(\xi_0 B \dot{\otimes}_B X) \oplus (\xi_0 B \ddot{\otimes}_B X)$, the above observations show that for any $x_k \in A_{\iota(k)}$ for $k = 1, \dots, m$ with $\iota \in \mathcal{I}_m$, the subspace $\psi_t(x_1 \cdots x_m)\Gamma$ is contained in $\Gamma_1 + X(\ell, 2)^\circ \dot{\otimes}_B X + X(\ell, 2)^\circ \ddot{\otimes}_B X$ when $\iota(1) = 1$, and in $\Gamma_2 + X(\ell, 1)^\circ \dot{\otimes}_B X + X(\ell, 1)^\circ \ddot{\otimes}_B X$ when $\iota(1) = 2$. This implies that Γ satisfies the freeness condition. \square

3.2. Case of countably many free components. Let \mathcal{I} be a general countable set and let $\mathfrak{A} = \star_{B, i \in \mathcal{I}} A_i$ and $(A, E) = \star_{B, i \in \mathcal{I}} (A_i, E_B^{A_i})$ be as in Theorem A. We set $c_0 := c_0(\mathcal{I})$ and $\mathcal{K} := \mathbb{K}(\ell^2(\mathcal{I}))$.

Proposition 3.6. *With the notation above, there exist nondegenerate conditional expectations $\sum_i E_B^{A_i} : \sum_i A_i \rightarrow c_0 \otimes B$ and $E_{c_0} \otimes \text{id}_B : \mathcal{K} \otimes B \rightarrow c_0 \otimes B$. If we set $\tilde{\mathfrak{A}} := (\sum_i A_i) \star_{c_0 \otimes B} (\mathcal{K} \otimes B)$ and $(\tilde{A}, \tilde{E}) := (\sum_i A_i, \sum_i E_B^{A_i}) \star_{c_0 \otimes B} (\mathcal{K} \otimes B, E_{c_0} \otimes \text{id}_B)$, then there exist isomorphisms $\pi : \tilde{\mathfrak{A}} \rightarrow \mathcal{K} \otimes \mathfrak{A}$ and $\pi_{\text{red}} : \tilde{A} \rightarrow \mathcal{K} \otimes A$ such that the following diagram*

$$\begin{array}{ccc} \tilde{\mathfrak{A}} & \xrightarrow{\pi} & \mathcal{K} \otimes \mathfrak{A} \\ \tilde{\lambda} \downarrow & & \downarrow \text{id}_{\mathcal{K}} \otimes \lambda \\ \tilde{A} & \xrightarrow{\pi_{\text{red}}} & \mathcal{K} \otimes A \end{array}$$

commutes, where $\tilde{\lambda}$ is the canonical surjection.

Proof. Since the proof in the case when \mathcal{I} is finite is essentially same as (and easier than) the case when $\mathcal{I} = \mathbb{N}$, we may and do assume that $\mathcal{I} = \mathbb{N}$. Let $\{e_{ij}\}_{i,j \geq 1}$ be the system of matrix units for the canonical basis $\{\delta_i\}_{i \geq 1}$ of $\ell^2(\mathbb{N})$, and set $f_n := e_{nn}$. We realize $\sum_{n \geq 1} A_n$ and $c_0 \otimes B$ inside $\mathcal{K} \otimes A$ as

$$\sum_{n \geq 1} A_n = C^*\{f_n \otimes a \mid n \geq 1, a \in A_n\} \quad \text{and} \quad c_0 \otimes B = C^*\{f_n \otimes b \mid n \geq 1, b \in B\}.$$

Consider two conditional expectations $\sum_n E_B^{A_n} : \sum_n A_n \rightarrow c_0 \otimes B$ and $E_{c_0} \otimes \text{id}_B : \mathcal{K} \otimes B \rightarrow c_0 \otimes B$ defined by $(\sum_n E_B^{A_n})(f_i \otimes a) = f_i \otimes E_B^{A_i}(a)$ and $(E_{c_0} \otimes \text{id}_B)(e_{ij} \otimes b) = \delta_{i,j} f_i \otimes b$ for $i, j \in \mathbb{N}$, $a \in A_i$ and $b \in B$. Set $\tilde{\mathfrak{A}} := (\sum_{n \geq 1} A_n) \star_{c_0 \otimes B} (\mathcal{K} \otimes B)$ and $(\tilde{A}, \tilde{E}) := (\sum_{n \geq 1} A_n, \sum_n E_B^{A_n}) \star_{c_0 \otimes B} (\mathcal{K} \otimes B, E_{c_0} \otimes \text{id}_B)$ and let $\tilde{\lambda} : \tilde{\mathfrak{A}} \rightarrow \tilde{A}$ be the canonical surjection.

The inclusion maps $\sum_n A_n \hookrightarrow \mathcal{K} \otimes \mathfrak{A}$ and $\mathbb{K} \otimes B \hookrightarrow \mathcal{K} \otimes \mathfrak{A}$ induce a $*$ -homomorphism $\pi : \tilde{\mathfrak{A}} \rightarrow \mathcal{K} \otimes \mathfrak{A}$. For any $n, i, j \in \mathbb{N}$, $a \in A_n$ and $b, c \in B$, one has $e_{ij} \otimes bac = \pi(e_{in} \otimes b)\pi(f_n \otimes a)\pi(e_{nj} \otimes c) \in \pi(\tilde{\mathfrak{A}})$. Since $[BA_nB] = A_n$ holds, π is surjective. Define $\sigma_n : A_n \rightarrow \tilde{\mathfrak{A}}$ by $\sigma_n(bac) = (e_{1n} \otimes b)(f_n \otimes a)(e_{n1} \otimes c)$ for $a \in A_n$ and $b, c \in B$. We then obtain $\sigma = \bigstar_{n \geq 1} \sigma_n : \mathfrak{A} \rightarrow \tilde{\mathfrak{A}}$. Define $\tilde{\sigma} : \mathcal{K} \otimes \mathfrak{A} \rightarrow \tilde{\mathfrak{A}}$ by $\tilde{\sigma}(e_{ij} \otimes bac) = (e_{i1} \otimes b)\sigma(a)(e_{1j} \otimes c)$ for $a \in A$ and $i, j \geq 1$. Then, it is easy to see that $\tilde{\sigma} \circ \pi = \text{id}_{\tilde{\mathfrak{A}}}$, and hence π is bijective.

We next see that $(\text{id}_{\mathcal{K}} \otimes \lambda) \circ \pi$ and $\tilde{\lambda} \circ \tilde{\sigma}$ factor through $\tilde{\lambda} : \tilde{\mathfrak{A}} \rightarrow \tilde{A}$ and $\text{id}_{\mathcal{K}} \otimes \lambda : \mathcal{K} \otimes \mathfrak{A} \rightarrow \mathcal{K} \otimes A$, respectively. Note that $(\sum_n A_n)^\circ$ and $(\mathcal{K} \otimes B)^\circ$ are generated by $\mathcal{S}_1 = \{f_n \otimes a \mid a \in A_n^\circ, n \geq 1\}$ and $\mathcal{S}_2 = \{e_{ij} \otimes b \mid i, j \geq 1, i \neq j, b \in B\}$ as normed spaces, respectively. We represent $\mathcal{K} \otimes A$ on the Hilbert B -module $\ell^2(\mathbb{N}) \otimes L^2(A, E)$ faithfully and show that $\mathbb{C}\delta_1 \otimes \xi_E B$ satisfies the assumption of Lemma 2.7 for \mathcal{S}_1 and \mathcal{S}_2 . Let $m \geq 1$, $x_1, \dots, x_m \in \mathcal{S}_1$ and $y_1, \dots, y_m \in \mathcal{S}_2$ be arbitrarily given. If $x_1 y_2 \cdots x_m y_m$ is a nonzero element, then it should be of the form

$$(f_{\iota(1)} \otimes a_1)(e_{\iota(1)\iota(2)} \otimes b_1) \cdots (f_{\iota(m)} \otimes a_m)(e_{\iota(m)\iota(m+1)} \otimes b_m) = e_{\iota(1)\iota(m+1)} \otimes (a_1 b_1 \cdots a_m b_m)$$

for some $\iota \in \mathcal{I}_{m+1}$, $a_k \in A_{\iota(k)}^\circ$ and $b_k \in B$ for $1 \leq k \leq m$. Clearly, this implies that $(x_1 y_1 \cdots x_m y_m)(\delta_1 \otimes \xi_E B) \perp \delta_1 \otimes \xi_E B$. A similar assertion holds for $x_1 y_1 \cdots y_m x_{m+1}$, $y_0 x_1 \cdots x_m y_m$ and $y_0 x_1 \cdots y_m x_{m+1}$ for any $x_{m+1} \in \mathcal{S}_1$ and $y_0 \in \mathcal{S}_2$. Therefore, Lemma 2.7 guarantees that $(\text{id}_{\mathcal{K}} \otimes \lambda) \circ \pi$ factors through $\tilde{\lambda}$.

Similarly, representing \tilde{A} on $L^2(\tilde{A}, \tilde{E})$ faithfully we observe that $\xi_{\tilde{E}}(c_0 \otimes B)$ satisfies the freeness condition for $\mathcal{S}'_n := BA_n^\circ B$, $n \geq 1$. Indeed, for $m \geq 1$, $\iota \in \mathcal{I}_m$, $y_k \in A_{\iota(k)}^\circ$, and $b_k, c_k \in B$, we have

$$\sigma((b_1 y_1 c_1) \cdots (b_m y_m c_m)) = (e_{\iota(1)} \otimes b_1)(f_{\iota(1)} \otimes y_1)(e_{\iota(1)\iota(2)} \otimes c_1 b_2) \cdots (f_{\iota(m)} \otimes y_m)(e_{\iota(m)} \otimes c_m),$$

which belongs to $\ker \tilde{E}$. Thus, $\tilde{\lambda} \circ \sigma : \mathfrak{A} \rightarrow \tilde{A}$ factors through $\lambda : \mathfrak{A} \rightarrow A$, which implies that $\tilde{\lambda} \circ \tilde{\sigma}$ factors through $\text{id}_{\mathcal{K}} \otimes \lambda : \mathcal{K} \otimes \mathfrak{A} \rightarrow \mathcal{K} \otimes A$. \square

The following general fact is well-known (see, e.g. [1, Proposition 17.8.7]).

Proposition 3.7. *Let \mathcal{K} be as above and let $\iota : \mathcal{K} \hookrightarrow \mathbb{B}(\ell^2(\mathcal{I}))$ be the inclusion map. Fix a minimal projection $e \in \mathcal{K}$. For any separable C^* -algebras \mathcal{A} and \mathcal{B} , the mapping $\mathbb{E}(\mathcal{A}, \mathcal{B}) \ni (X, \phi, F) \mapsto (\mathcal{K} \otimes X, \lambda_{\mathcal{K}} \otimes \phi, 1_{\mathcal{K}} \otimes F) \in \mathbb{E}(\mathcal{K} \otimes \mathcal{A}, \mathcal{K} \otimes \mathcal{B})$ induces an isomorphism $\tau : KK(\mathcal{A}, \mathcal{B}) \rightarrow KK(\mathcal{K} \otimes \mathcal{A}, \mathcal{K} \otimes \mathcal{B})$. The inverse of τ is given by the mapping $\mathbb{E}(\mathcal{K} \otimes \mathcal{A}, \mathcal{K} \otimes \mathcal{B}) \ni (Y, \psi, G) \mapsto (Y \otimes_{\iota \otimes \lambda_{\mathcal{B}}} (\ell^2(\mathcal{I}) \otimes \mathcal{B}), (\psi \otimes_{\iota \otimes \lambda_{\mathcal{B}}} 1) \circ \sigma, G \otimes_{\iota \otimes \lambda_{\mathcal{B}}} 1) \in \mathbb{E}(\mathcal{A}, \mathcal{B})$, where $\sigma(a) = e \otimes a$ for $a \in \mathcal{A}$.*

We are now ready to prove Theorem A and Corollary B.

Proof of Theorem A and Corollary B. We use the notation in the proof of Proposition 3.6. By Theorem 3.4 and Proposition 3.6, there exists $\beta \in KK(\mathcal{K} \otimes A, \mathcal{K} \otimes \mathfrak{A})$ such that $\beta \circ (\text{id}_{\mathcal{K}} \otimes \lambda) = \text{id}_{\mathcal{K} \otimes \mathfrak{A}}$ and $(\text{id}_{\mathcal{K}} \otimes \lambda) \circ \beta = \text{id}_{\mathcal{K} \otimes A}$. Let τ be as in Proposition 3.7. We then have $\text{id}_{\mathfrak{A}} = \tau^{-1}(\text{id}_{\mathcal{K} \otimes \mathfrak{A}}) = \tau^{-1}(\beta \circ (\text{id}_{\mathcal{K}} \otimes \lambda)) = \tau^{-1}(\beta) \circ \lambda$ and $\text{id}_A = \tau^{-1}(\text{id}_{\mathcal{K} \otimes A}) = \tau^{-1}((\text{id}_{\mathcal{K}} \otimes \lambda) \circ \beta) = \lambda \circ \tau^{-1}(\beta)$. Thus, λ gives a KK -equivalence.

Moreover, by Theorem 3.4 and Proposition 3.7 again, $\tau^{-1}(\beta)$ is implemented by a Kasparov A - \mathfrak{A} bimodule whose “ C^* -correspondence part” is the direct sum of three C^* -correspondences of the form $(Y \otimes_D Z, \pi_Y \otimes_D 1_Z)$, where D is either $c_0 \otimes B$, $\sum_i A_i$ or $\mathcal{K} \otimes B$. Thus, if A_i is nuclear for every $i \in \mathcal{I}$, then $\text{id}_{\mathfrak{A}} = \tau^{-1}(\beta) \circ \lambda$ is also implemented a Kasparov bimodule consisting of a nuclear C^* -correspondence (see the remark just after Theorem 2.5), and hence \mathfrak{A} is K -nuclear. \square

Remark 3.8. Theorem A generalizes the previous K -amenability results for amalgamated free products of amenable discrete (quantum) groups [11] and [20]. However, we should remark that our result does not imply Pimsner’s result that K -amenability is closed under amalgamated free products. Similarly, Corollary B does not imply that K -nuclearity is closed under amalgamated free products (even for plain free products). The latter seems a next interesting question in the direction.

4. SIX-TERM EXACT SEQUENCES

Let $(A, E) = (A_1, E_B^{A_1}) \star (A_2, E_B^{A_2})$ is as in Theorem 3.4. We denote by $i_k : B \rightarrow A_k$ and $j_k : A_k \rightarrow A, k = 1, 2$ the inclusion maps. As we mentioned in the introduction, our KK -equivalence and K -nuclearity results with Thomsen's result [17] imply the following:

Corollary 4.1. *With the notation above, there is a cyclic six-term exact sequence*

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{(i_{1*}, i_{2*})} & K_0(A_1) \oplus K_0(A_2) & \xrightarrow{j_{1*} - j_{2*}} & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \xleftarrow{j_{1*} - j_{2*}} & K_1(A_1) \oplus K_1(A_2) & \xleftarrow{(i_{1*}, i_{2*})} & K_1(B) \end{array}$$

If A_1 and A_2 are further assumed to be nuclear, then for any separable C^* -algebras D there is a cyclic exact sequence

$$\begin{array}{ccccc} KK(B, D) & \xleftarrow{i_1^* - i_2^*} & KK(A_1, D) \oplus KK(A_2, D) & \xleftarrow{j_1^* + j_2^*} & KK(A, D) \\ \downarrow & & & & \uparrow \\ KK(A, SD) & \xrightarrow{j_1^* + j_2^*} & KK(A_1, SD) \oplus KK(A_2, SD) & \xrightarrow{i_1^* - i_2^*} & KK(B, SD) \end{array}$$

Note that the second exact sequence of KK -groups is new even in the full case. We also obtain the next corollary from Theorem A and [4].

Corollary 4.2. *With the notation above, suppose that B is a direct sum of finite dimensional C^* -algebras. Then, for any separable C^* -algebra D there are two cyclic exact sequences:*

$$\begin{array}{ccccc} KK(D, B) & \xrightarrow{(i_{1*}, i_{2*})} & KK(D, A_1) \oplus KK(D, A_2) & \xrightarrow{j_{1*} - j_{2*}} & KK(D, A) \\ \uparrow & & & & \downarrow \\ KK(SD, A) & \xleftarrow{j_{1*} - j_{2*}} & KK(SD, A_1) \oplus KK(SD, A_2) & \xleftarrow{(i_{1*}, i_{2*})} & KK(SD, B) \\ KK(B, D) & \xleftarrow{i_1^* - i_2^*} & KK(A_1, D) \oplus KK(A_2, D) & \xleftarrow{j_1^* + j_2^*} & KK(A, D) \\ \downarrow & & & & \uparrow \\ KK(A, SD) & \xrightarrow{j_1^* + j_2^*} & KK(A_1, SD) \oplus KK(A_2, SD) & \xrightarrow{i_1^* - i_2^*} & KK(B, SD) \end{array}$$

Finally, we would like to point out that a similar result holds for HNN extensions. We refer the reader to [18, 19] for HNN extensions of C^* -algebras. The next corollary follows from “the C^* -version of Proposition 3.1”, Proposition 3.3 and Proposition 4.2 in [19] and Theorem A.

Corollary 4.3. *Let $B \subset A$ be unital inclusion of separable C^* -algebras with nondegenerate conditional expectation $E : A \rightarrow B$, and $\theta : B \rightarrow A$ be an injective $*$ -homomorphism whose image is the range of a conditional expectation $E_\theta : A \rightarrow \theta(B)$. Then, the full HNN-extension $A \star_B^{\text{univ}} \theta$ and the reduced one $(A, E) \star_B (\theta, E_\theta)$ are KK -equivalent via the canonical surjection, and there is a six-term exact sequence:*

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{(\theta_* - \iota_{B*})} & K_0(A) & \xrightarrow{\iota_{A*}} & K_0((A, E) \star_B (\theta, E_\theta)) \\ \uparrow & & & & \downarrow \\ K_1((A, E) \star_B (\theta, E_\theta)) & \xleftarrow{\iota_{A*}} & K_1(A) & \xleftarrow{(\theta_* - \iota_{B*})} & K_1(B) \end{array}$$

Here $\iota_B : B \rightarrow A$ and $\iota_A : A \rightarrow (A, E) \star_B (\theta, E_\theta)$ are inclusion maps. Further assume that A is nuclear. Then, these HNN-extensions are K -nuclear.

Remark 4.4. Using Proposition 3.7 we can generalize the results in this section to amalgamated free products and HNN extensions of countably many C^* -algebras and countably many injective $*$ -homomorphisms, respectively. Such generalizations for HNN extensions include Pimsner–Voiculescu’s six-term exact sequence for crossed products by free groups ([14, 15]) as special cases (see also [19]).

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